Contents 41

Hypothesis Testing

Learning outcomes

By studying this Workbook you will learn how to apply statistical techniques to test the validity, on the basis of available evidence, of a given hypothesis. For example, a motor engineer may be interested in testing the expected life of a given set of tyres ("the mean life is 2,000 miles") against an alternative ("the mean life is less than 2,000 miles"). You will learn about techniques which will enable you to answer such questions.

This Workbook will introduce you to the basic ideas of hypothesis testing in a non-mathematical way by using a problem solving approach to highlight the concepts as they are needed.

Once you have learned how to apply the basic ideas, you will be capable of applying hypothesis testing to a very wide range of practical problems and learning about methods of hypothesis testing which are not covered in this Workbook.

Statistical Testing 41.1

Introduction

If you are applying statistics to practical problems in industry, you may find that much of your work is concerned with making decisions concerning populations and population parameters on the basis of available evidence. For example you may be asked to decide whether one production process is preferable to another or whether to repair or continue to use a machine that is producing a certain proportion of defective components. In order to make such decisions, you will find that you have to make certain assumptions which will determine the statistical tools that you may legitimately use. Any assumptions made may or may not be true but you must always be sure of your grounds for using a given statistical tool. Effectively you will find that you will be asked to decide which of two statements, each called an hypothesis, is the more likely to be true. Note the choice of words. You should be clear from the outset that the statistical tools you will study here will not allow you to prove anything, but they will allow you to measure the strength of the evidence against the hypothesis.

Prerequisites

Before starting this Section you should ...

- understand the term 'sample'
- be able to differentiate between statements which are a matter of opinion and those which are of a numerical nature and as such can be challenged

Learning Outcomes

On completion you should be able to ...

- understand what is meant by the terms hypothesis and hypothesis testing
- understand the what is meant by the terms one-tailed test and two-tailed test
- understand what is meant by the terms type I error and type II error
- understand the term level of significance
- apply a variety of statistical tests to problems based in engineering

1. Types of statements

Almost every time we read a magazine or newspaper we see claims made by manufacturers about their products. Such claims can take many forms, they may for example be subjective:

'Luxcar, makers of the best luxury cars'

'Burnol, the finest fuel you can buy'

'ConstructAll, designers of beautiful buildings'

Such claims do not need to be backed up by facts and figures, they are a matter of opinion.

Many claims do contain information which is open to question and can be investigated statistically:

'the expected life of these tyres is 20,000 miles'

'on average, low energy light bulbs can be expected to last at least 8000 hours'

'average bottle contents 330 ml.'

The validity of claims which contain information of a numerical nature can often be investigated by taking random samples of the objects or quantities in question and investigating the likelihood that a statement or hypothesis concerning them is true.

As stated in the introduction, it should be noted that hypothesis testing can never prove that a statement is either true or false, it can only give a measure of the truth or otherwise of a given statement. Statements which are investigated statistically are normally called hypotheses and we usually try to establish a pair of hypotheses, called a **null hypothesis** and an **alternative hypothesis** and then investigate how the evidence that we have supports one hypothesis more than the other. For example, a demolition engineer might be interested in the burn rate of fuses connected to explosive devices and on the basis of experience hypothesize that the mean burn rate (say μ) is 600 mm/sec. A colleague may disagree and claim that the mean burn rate is greater than 600 mm/sec.

We can describe this situation by setting up the null hypothesis:

$H_0: \ \mu = 600$

and test this against the alternative hypothesis:

$H_1: \ \mu > 600$

2. Types of errors

Since we cannot be 100% sure that a hypothesis is true or false it is possible that:

(a) a correct hypothesis will be rejected;

(b) a false hypothesis will be accepted.

Rejecting a correct hypothesis is called a Type I error and accepting a false hypothesis is called a Type II error.

By working in a logical manner and developing a set of rules or guide-lines, it is possible to minimise the occurrence of such errors.

This will introduce you to the basic ideas of hypothesis testing in a non-mathematical way by using a problem solving approach to highlight the concepts as they are needed.

Once you have learned how to apply the basic ideas, you will be capable of applying hypothesis testing to a very wide range of practical problems and learning about methods of hypothesis testing which are not covered in this Workbook.

Tests Concerning a Single Sample

41.2

Introduction

This Section introduces you to the basic ideas of hypothesis testing in a non-mathematical way by using a problem solving approach to highlight the concepts as they are needed. We only consider situations involving a single sample.

In Section 41.3 we will introduce you to situations involving two samples and while the basic ideas will follow through, their practical application is a little more complex than that met in this Workbook. However, once you have learned how to apply the basic ideas of hypothesis testing covered in this Workbook, you should be capable of applying hypothesis testing to a very wide range of practical problems and learning about methods of hypothesis testing which are not covered here.

Prerequisites

Before starting this Section you should ...

- be familiar with the results and concepts met in the study of probability
- be familiar with a range of statistical distributions
- understand the term hypothesis
- understand the concepts of Type I error and Type II error

Learning Outcomes

On completion you should be able to ...

- apply the ideas of hypothesis testing to a range of problems underpinned by elementary statistical distributions and involving only a single sample.

1. Tests of proportion

Problem 1

SwitchRight, a manufacturer of engine management systems requires its supplier of control modules to supply modules with at least 99% complying with their specification. The quality control operators at SwitchRight check a random sample of 1000 control modules delivered to SwitchRight and find that 985 match the specification. Does this result imply that less than 99% of the control modules supplied do not match SwitchRight's specification?

Analysis

Firstly, we set up two hypotheses concerning the control modules. The first hypothesis, called the null hypothesis is denoted by

$H_0:$ 99% of the control modules match SwitchRight's specification.

The second hypothesis, called the alternative hypothesis and is denoted by

$H_1:$ less than 99% of the control modules match SwitchRight's specification.

The alternative hypothesis is essentially saying that in this case, that SwitchRight cannot rely on its supplier of control modules supplying delivering batches of modules where 99% match SwitchRight's specification.

Secondly, we describe the random sample from a statistical point of view, that is we find a statistical distribution which describes the behaviour of the sample. Suppose that X is the number of control modules in a random sample of 1000 matching SwitchRight's specification.

We assume that the control modules are independent and that for each module the specification is either matched or it isn't. Under these conditions, X has a binomial distribution and the problem can be summarised as follows:

$X \sim B(1000, p)$

$H_0: \quad p = 0.99 \qquad H_1: \quad p < 0.99$

Thirdly, we set up a mechanism to enable us to make a decision between the two hypotheses. This is done by assuming that H_0 **is correct** until we can show otherwise.

Given that H_0 is correct we can calculate the mean μ and the standard deviation σ of the distribution as follows:

$$\mu = np = 1000 \times 0.99 = 990$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{1000 \times 0.99 \times 0.01} = 3.15$$

Notice that

(a) $np > 5$ and (b) $n(1-p) > 5$

so that we can use the normal approximation to the binomial distribution, that is

$$B(1000, 0.99) \approx N(990, 3.15^2)$$

The sample value obtained is 985 and we now assess how close 985 is to the expected result of 990 by defining a remote left tail (in this case) of the normal distribution and asking if the number 985 occurs in the left tail of the distribution or in the main body of the distribution.

In practice, we use the tail(s) of the standard normal distribution and convert a problem involving the distribution $N(\mu, \sigma^2)$ into one involving the distribution $N(0, 1)$. Diagrammatically the situation can be represented as shown below:

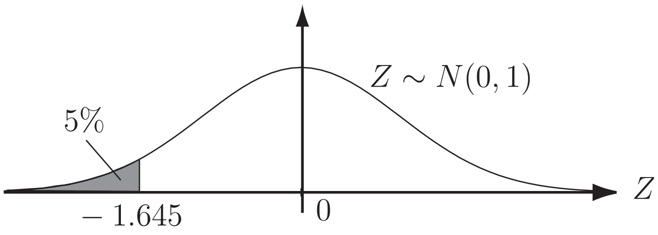


Figure 1

In general, the tails of a distribution can be defined to occupy any proportion of the distribution that we wish, the proportions chosen are usually taken as either 5% or 1%.

Given this information and a set of tables for the standard normal distribution we can assign values to the limits defining the tails.

Throughout this Workbook we shall use the 5% proportion to define the tail(s) of a distribution unless otherwise stated.

In the case we have here, the alternative hypothesis states that p is **less** than 0.99. Because of this we use only one tail occupying a **total** of 5% of the distribution.

To discover where the number 985 lies within the distribution (tail or main body) we standardise 985 with respect to the normal distribution $N(990, 3.15^2)$ in the usual way (see HELM 39). The calculation is:

$$P(X \leq 985) = P\left(Z \leq \frac{985.5 - 990}{3.15}\right) = P(Z \leq -1.43)$$

Notice that 985.5 is used and not 985. This because we are using a *continuous* normal distribution to approximate a *discrete* binomial distribution and so

$$P(X = 985) \approx P(984.5 \leq X \leq 985.5)$$

the right-hand side being calculated from the normal distribution.

The number -1.43 is greater than (to the right of) -1.645 and so the number 985 occurs in the main body of the distribution not in the left tail. This suggests that the evidence does not support the claim that the number of control modules supplied meeting SwitchRight's specification is different from 99%. Essentially, we accept the null hypothesis since we do not have the evidence necessary to reject it. Note that this result does not **prove** that the claim is true.

Before looking at similar problems, we will look at the possible ways of defining the tails of the standard normal distribution. As stated previously, we shall, in these notes, always use a total of 5% for the tail or tails of a distribution.

We say that we are making a decision at the **5% level of significance**.

The situation is represented by the following three figures:

(1) **Hypotheses**:-

$H_0: \quad p = p_0$

$H_1: \quad p \neq p_0$

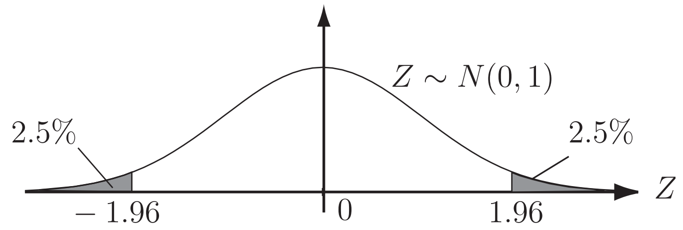


Figure 2

(2) **Hypotheses**:-

$H_0: \quad p = p_0$

$H_1: \quad p > p_0$

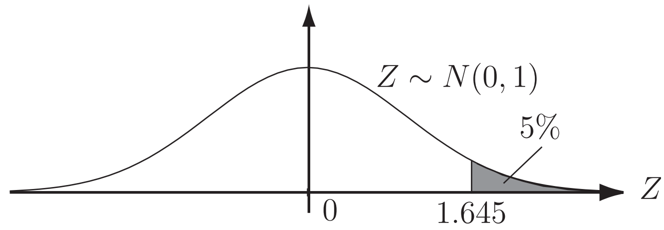


Figure 3

(3) **Hypotheses**:-

$H_0: \quad p = p_0$

$H_1: \quad p < p_0$

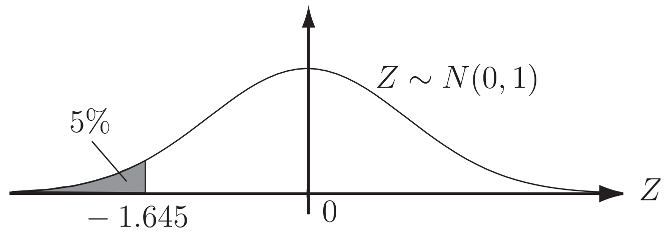


Figure 4

The values ± 1.96, $+1.645$ and -1.645 are easily obtained from the standard normal table (Table 1) given at the end of this Workbook. The appropriate lines from the table are reproduced on the following page for ease of reference. Note that it is sometimes advisable to be 99% sure (rather than 95% sure) of either correctly accepting or rejecting a null hypothesis. In this case we say that we are working at the 1% level of significance. The situation diagrammatically is exactly the same as the one shown above except that the 5% tail areas become 1% and the 2.5% areas become 0.5%.

The corresponding values of Z are ± 2.58, $+2.33$ and -2.33 depending on whether a one-tailed or a two-tailed test is being performed.

Particular note must always be taken of the form of the hypotheses and the corresponding test, one-tailed or two-tailed.

Extracts from the normal probability integral table

Case 1 - 5% level of significance

$Z=\frac{X-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.6	.4452	4463	4474	4485	4495	4505	4515	4525	4535	4545
1.9	.4713	4719	4726	4732	4738	4744	4750	4756	4762	4767

Case 2 - 1% level of significance

$Z=\frac{X-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.3	.4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.5	.4938	4940	4941	4943	4945	4946	4948	4949	4951	4952

We shall now look at a problem which is similar in type to Problem 1 and solve it using the ideas discussed in the analysis of that problem.

Problem 2

The Head of Quality Control in a foundry claims that the castings produced in the foundry are 'better than average.' In support of this claim he points out that of a random sample of 60 castings inspected, 59 passed. It is known that the industry average percentage of castings passing quality control inspections is 90%. Do these results support the Head's claim?

Analysis

Let X denote the number of castings passing the quality control inspection from the sample of 60. Assuming that a casting either passes or fails the inspection process, we can assume that X follows the binomial distribution

$$X \sim B(60, p)$$

where p is the probability that a casting passes the inspection.

The null hypothesis H_0, is that the probability that a casting passes the inspection is the same as the industry average. The alternative hypothesis H_1, is that the Head of Quality Control is correct in his claim that castings produced in his foundry have a greater chance of passing the inspection. The problem can be summarised as:

$$X \sim B(60, p)$$

$$H_0 : \quad p = 0.90 \qquad\qquad H_1 : \quad p > 0.90$$

The form of the alternative hypothesis dictates that we do a one-tailed test.

If H_0 is correct we can calculate the mean and standard deviation of the binomial distribution above and, assuming that the appropriate condition are met, use the normal distribution with the same mean and standard deviation to solve the problem. The calculations are:

$$\mu = np = 60 \times 0.90 = 54$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{60 \times 0.90 \times 0.10} = 2.32$$

Notice that

(a) $np > 5$ and (b) $n(1-p) > 5$

so that we can use the normal approximation to the binomial distribution, that is

$$B(60, 0.90) \approx N(54, 2.32^2)$$

In order to make a decision, we need to know whether or not the value 59 is in the remote tails of the distribution or in the main body. Recall that the hypotheses are:

$$H_0: \quad p = 0.90 \qquad H_1: \quad p > 0.90$$

so that we must do a one-tailed test with a critical value of $Z = 1.645$.

The calculation is:-

$$P(X \geq 59) = P\left(Z \geq \frac{58.5 - 54}{2.32}\right) = P(Z \geq 1.94)$$

The situation is represented by the following figure.

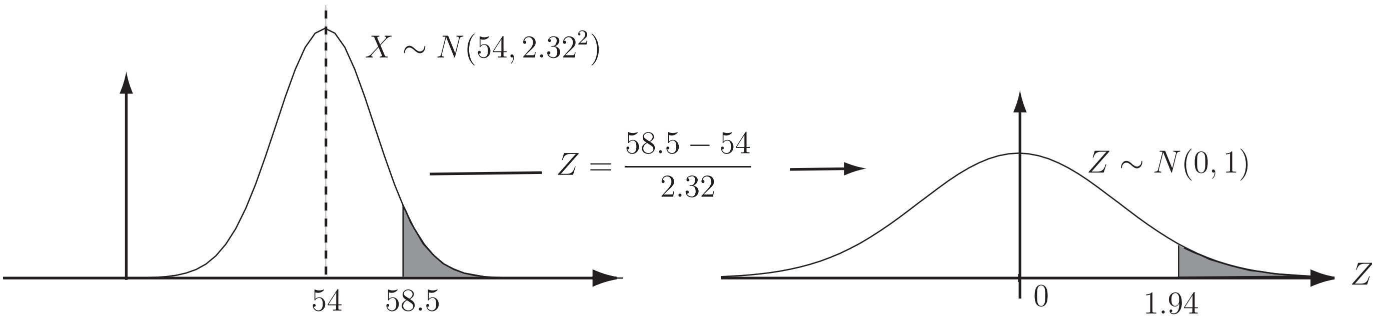


Figure 5

Since $1.94 > 1.645$, the result is significant at the 5% level and so we reject the null hypothesis. The evidence suggests that we accept the alternative hypothesis that, at the 5% level of significance, the Head of Quality Control is making a justified claim.

Task

A firm manufactures heavy current switch units which depend for their correct operation on a relay. The relays are provided by an outside supplier and out of a random sample of 150 relays delivered, 140 are found to work correctly. Can the relay manufacturer justifiably claim that at least 90% of the relays provided will function correctly?

Your solution

Answer

Let X represent the number of relays working correctly. The required hypotheses are:

$$X \sim B(150, p) \qquad H_0: \quad p = 0.90 \qquad H_1: \quad p > 0.90$$

We perform a one-tailed test with critical value $Z = 1.645$. The necessary calculations are:

$$\mu = np = 150 \times 0.90 = 135$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{150 \times 0.90 \times 0.10} = 3.67$$

Since $np > 5$ and $n(1-p) > 5$, we can use the normal approximation to the binomial distribution. We approximate $B(150, 0.90) \approx N(135, 3.67^2)$. Hence:

$$P(X \geq 140) = P\left(Z = \frac{139.5 - 135}{3.67}\right) = P(Z \geq 1.23)$$

Since $1.23 < 1.645$ we cannot reject the null hypothesis at the 5% level of significance.

There is insufficient evidence to support the manufacturer's claim that at least 90% of the relays provided will function correctly.

2. Tests for population means

Tests concerning a single mean

Introduction

In cases where tests involving measurements are performed, it is often possible to statistically hypothesize about the results. Suppose that the boiling point of a particular coolant used in car engines is claimed by a manufacturer to be $110°$C. Further suppose that a series of accurate measurements made in a laboratory using 8 random samples of the coolant are recorded as:

$$110.2°,\ 110.3°,\ 110.1°,\ 109.8°,\ 109.9°,\ 110.0°,\ 110.4°,\ 110.1°,$$

The mean of these results is $110.1°$C.

It is reasonable to ask whether, on the basis of the results obtained, we may claim that the boiling point of the coolant is greater than the assumed true boiling point of $110°$C. We will return to this problem later in this Workbook after looking at some general results.

General results

In general terms, we need to make predictions, based on calculation, about the parameters of the population from which the random sample is drawn. As illustrated above we calculate the sample mean $\bar{x}$. The statistical tests used to answer the above question depend on whether the variance of the population is known or not.

Case (i) - Population variance known

Firstly we form the null hypothesis that there is no difference between the true population mean μ and the theoretical value μ_0. That is:

$$H_0: \ \mu = \mu_0$$

Secondly we consider drawing samples of size n from the population. If n is **large** (say $n \geq 30$) then, because of the central limit theorem, we can often assume that the sample means approximately follow a normal distribution with mean μ and standard deviation (standard error of the mean) σ_n given by

$$\sigma_n = \frac{\sigma}{\sqrt{n}}$$

It follows that

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

has a standard normal distribution when the null hypothesis is true. That is, when $\mu = \mu_0$, $Z \sim N(0,1)$.

We may now set up an alternative hypothesis which can take one of the three forms:

$$\begin{array}{l} H_1: \ \mu \neq \mu_0 \\ H_1: \ \mu > \mu_0 \\ H_1: \ \mu < \mu_0 \end{array}$$

depending on the form of deviation from the null hypothesis for which we wish to test. Then we will reject the null hypothesis at the 5% level of significance if

$\lvert Z\rvert > 1.96$	for a two-tailed test
$Z > 1.645$	for a (right) one-tailed test
$Z < -1.645$	for a (left) one-tailed test

In each case we reject H_0 in favour of the alternative hypothesis when Z lies in the remote tail of the standard normal distribution.

Example 1

Dishwasher powder is poured into the cartons in which it is sold by an automatic dispensing machine which is set to dispense 3 kg of powder into each carton. In order to check that the dispensing machine is working to an acceptable standard (i.e. does not need adjustment), a production engineer takes a random samples of 40 cartons and weighs them. It is found that the mean weight of the sample is 3.005 kg. It is known that the dispensing machine operates with a variance of 0.015^2 kg^2 and that the manufacturer of the powder is willing to rely on a 5% level of significance. Does the sample provide the engineer with sufficient evidence that the true mean is not 3.00 kg and so the machine requires adjustment?

Solution

Given that the dispensing machine can over-fill or under-fill the containers, the null and alternative hypotheses are:

$$H_0: \quad \mu = 3 \qquad H_1: \quad \mu \neq 3$$

Since the sample size is large (≥ 30) and we can regard the population as infinite but with a known variance, we can calculate the relevant value of the test statistic Z by using the formula:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Hence, in this case:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{3.005 - 3}{0.015/\sqrt{40}} = 2.108$$

and since we are performing a two-tailed test at the 5% level of significance and have found that $|Z| > 1.96$, that is, Z is outside the range $[-1.96, 1.96]$, we must reject the null hypothesis and conclude that the machine is not operating acceptably and needs adjustment.

Case (ii) - Population variance unknown

We have exactly the same situation as that described in Case (i) but do not know the value of the population variance σ^2. Therefore we estimate it using

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

and calculate the test statistic

$$T = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}.$$

However, because we are now dividing by an estimate, which is itself random, this test statistic does not have a standard normal distribution under the null hypothesis. Instead it has a distribution called **Student's t-distribution** on $n-1$ degrees of freedom. The number of degrees of freedom is the same as that which we have already seen when we looked at the χ^2 distribution in connection with sample variances in Workbook 40. So, for example, instead of comparing Z with ± 1.96 for a two-sided test at the 5% level, when σ^2 is known, we compare T with a value from the t-distribution which depends on the sample size through the number of degrees of freedom. The t-distribution is symmetric, centred at zero and, for all but very small numbers of degrees of freedom, has a shape similar to that of a standard normal distribution but with a larger variance. A table which gives the values which we need is provided at the back of this Workbook. For example, if we have a two-sided test at the 5% level of significance and a sample size $n = 15$, then the number of degrees of freedom is 14 and we compare $|T|$ with the upper 2.5% point which is 2.145.

Looking at the table and comparing it with the values for a standard normal distribution we can see that, as the number of degrees of freedom becomes large, the t-distribution gets closer to the standard normal distribution so that, for large samples, it makes little difference which we use. It is also true that, under most circumstances, even if we do not know that the distribution from which

data are drawn is normal, a t-test provides a good approximation when the sample size is reasonably large. In other circumstances, for example when normality cannot be assumed and the sample is small, we need to use other procedures, often non-parametric tests.
In summary we have the following.

Population	Variance	Sample size	Test
Normal	Known	Small	Normal (Z)
Normal	Known	Large	Normal (Z)
Normal	Unknown	Small	t
Normal	Unknown	Large	t but Z approximates
Not Normal	Either	Small	Non-parametric
Not Normal	Known	Large	Z approximates
Not Normal	Unknown	Large	Z and t approximate

Non-parametric testing is covered in HELM 45.

Example 2

The average useful life of a random sample of 33 similar calculator batteries made on a production line is found to be 99.5 hours continuous use. The sample variance is 18.49 hours^2. Test the null hypothesis that the population mean lifetime is 100 hours against the alternative that it is less. Use the 5% level of significance.

Solution

The null and alternative hypotheses are:

$$H_0: \ \mu = 100 \qquad H_1: \ \mu < 100$$

Our test statistic is

$$T = \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}}$$

In this case

$$\begin{aligned} T &= \frac{99.5 - 100.0}{\sqrt{18.49/33}} \\ &= -0.668 \end{aligned}$$

and the number of degrees of freedom is $n - 1 = 33 - 1 = 32$. The table does not give values for 32 degrees of freedom but it does give values for 30 degrees of freedom and for 40 and the values for 32 must be in between. The lower 5% points for 30 and 40 degrees of freedom are -1.697 and -1.684 respectively. Clearly our observed value of -0.668 is not significant and we do not have sufficient evidence to reject the null hypothesis that $\mu = 100$.

Task

Solve the problem given at the start of subsection 2 (page 11). Note the sample is small and you will have to estimate the population variance from the sample variance. Use the tabulated values of the t-distribution given at the end of this Workbook in conjunction with the appropriate number of degrees of freedom.

Your solution

Answer

The null and alternative hypotheses are:

$$H_0: \quad \mu = 110 \qquad H_1: \quad \mu > 110$$

The value of the sample variance is given by the formula

$$s^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{0.28}{7} = 0.004$$

The test statistic t is given by

$$t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{110.1-110}{\sqrt{0.04}/\sqrt{8}} = \frac{0.1\times\sqrt{8}}{0.2} = 1.414$$

At the 5% level of significance and using $8-1=7$ degrees of freedom, the value of $t_{\alpha,\nu}$ from tables is 1.895. Since $1.414 < 1.895$, we cannot reject the null hypothesis in favour of the alternative hypothesis. On the basis of the evidence available, we are not able to conclude that the boiling point of the coolant is greater than $110°$C.

General comments about tests concerning a population mean

(a) The sample mean $\bar{x}$ is often used as a test statistic when testing a hypothesis concerning a population mean μ.

(b) Even if the population distribution cannot be assumed to be normal, the distribution of sample means can often be assumed to be normal. This depends on the sample size.

(c) The tests described above sometimes require us to assume that the population variance is known. This is often unrealistic and we turn to the t-test to deal with cases where the population standard deviation is unknown and must be estimated from the data available.

General comments on the *t*-test

(a) The test only applies when the underlying distribution can be assumed to be normal.

(b) The test is used when the standard deviation of the parent population has to be estimated.

(c) As the sample size n get larger, the distribution approximates to the standard normal distribution.

(d) The distribution depends on the number of degrees of freedom, for a single sample or equal paired samples (see below), the number of degrees of freedom is always one less than the sample size.

Tests concerning paired data

Sometimes experimental data may be directly compared using an appropriate test. The following Example looks at experimental data concerning the throttle reaction times of two turbochargers fitted to an internal combustion engine.

Example 3

In order to test the hypothesis that two standard turbochargers A and B have the same throttle reaction times, a random sample of 7 cars were fitted with the turbochargers and the throttle reaction times measured. The results were as follows:

Car	1	2	3	4	5	6	7
Throttle Reaction time for $A; R1$	0.223	0.212	0.201	0.205	0.216	0.211	0.209
Throttle Reaction time for $B; R2$	0.208	0.207	0.203	0.204	0.205	0.202	0.206
$D = R1 - R2$	0.015	0.005	−0.002	0.001	0.011	0.009	0.003

Solution

Let D be the difference between the throttle reaction times of the two turbochargers. We assume that the distribution of D is normal. Our null hypothesis is that μ_D, the mean of the population of differences, is zero. We must decide between the two hypotheses

$$H_0 : \ \mu_D = 0 \qquad H_1 : \ \mu_D \neq 0$$

The alternative hypothesis here indicates that we perform a two-tailed test.
Let $\bar{d}$ be the sample mean of the seven observed differences. Then

$$\bar{d} = \frac{\sum d}{7} = \frac{0.042}{7} = 0.006$$

The sample variance of the differences is

$$s_d^2 = \frac{\sum(d-\bar{d})^2}{n-1} = \frac{0.000214}{6} = 3.5667 \times 10^{-5}$$

The value of the test statistic is

$$|t| = \frac{|\bar{d}-0|}{\sqrt{s_d^2/n}} = \frac{0.006}{\sqrt{3.5667 \times 10^{-5}/7}} = 2.658$$

The number of degrees of freedom is $7-1=6$ and the critical value from the table is 2.447. Since $2.658 > 2.447$ we reject H_0 at the 5% level and conclude that the evidence suggests that there is a difference in the throttle reaction times between the two turbochargers.

Task

Two different methods of analysis were used to determine the levels of impurity present in a particular aircraft quality aluminium alloy. Eight specimens were analysed using both methods. Does the available evidence suggest that both methods lead to the same results?

Alloy Specimen	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Test 1	1.24	1.23	1.24	1.20	1.21	1.22	1.23	1.22
Test 2	1.23	1.20	1.20	1.21	1.20	1.20	1.21	1.25
$D =$ Test1 $-$ Test2	0.01	0.03	0.04	-0.01	0.01	0.02	0.02	-0.03

Your solution

Answer

Let D be the difference between the two methods of analysis. We assume that the distribution of D is normal. Our null hypothesis is that μ_D, the mean of the population of differences, is zero. We must decide between the two hypotheses

$$H_0: \ \mu_D = 0 \qquad H_1: \ \mu_D \neq 0$$

The alternative hypothesis here indicates that we perform a two-tailed test.

Let $\bar{d}$ be the sample mean of the eight observed differences. Then

$$\bar{d} = \frac{\sum d}{8} = \frac{0.09}{8} = 0.01125$$

The sample variance of the differences is

$$s_d^2 = \frac{\sum (d - \bar{d})^2}{n-1} = \frac{0.0034875}{7} = 0.0004982$$

The value of the test statistic is

$$|t| = \frac{|\bar{d} - 0|}{\sqrt{s_d^2/n}} = \frac{0.01125}{\sqrt{0.0004982/8}} = 1.426$$

The number of degrees of freedom is $8 - 1 = 7$ and the critical value from the table is 2.306. Since $-2.306 < 1.426 < 2.306$ we do not reject H_0 at the 5% level and conclude that there is insufficient evidence to show that there is a difference between the two methods.

Tests Concerning Two Samples

41.3

Introduction

So far we have dealt with situations in which we either had a single sample drawn from a population, or paired data whose differences were considered essentially as a single sample.

In this Section we shall look at the situations occurring when we have two random samples each drawn from *independent* populations. While the basic ideas involved will essentially repeat those already met, you will find that the calculations involved are more complex than those already covered. However, you will find as before that calculations do follow particular routines. Note that in general the samples will be of different sizes. Cases involving samples of the same size, while included, should be regarded as special cases.

Prerequisites

Before starting this Section you should ...

- be familiar with the normal distribution, t-distribution, F-distribution and chi-squared distribution

Learning Outcomes

On completion you should be able to ...

- apply the ideas of hypothesis testing to a range of problems underpinned by a substantial range of statistical distributions and involving two samples of different sizes

1. Tests concerning two samples

Two independent populations each with a known variance

We assume that the populations are normally distributed. This may not always be true and you should note this basic assumption while studying this Section of the Workbook.

A standard notation often used to describe the populations and samples is:

Population	Sample
$X_1 \sim N(\mu_1, \sigma_1^2)$	$x_{11}, x_{12}, x_{13}, \cdots, x_{1n_1}$ with n_1 members.
$X_2 \sim N(\mu_2, \sigma_2^2)$	$x_{21}, x_{22}, x_{23}, \cdots, x_{2n_2}$ with n_2 members.

If you are not familiar with the double suffix notation used to represent the samples, simply remember that a random sample of size n_1 is drawn from $X_1 \sim N(\mu_1, \sigma_1^2)$ and a random sample of size n_2 is drawn from $X_1 \sim N(\mu_1, \sigma_1^2)$.

In diagrammatic form the populations may be represented as follows:

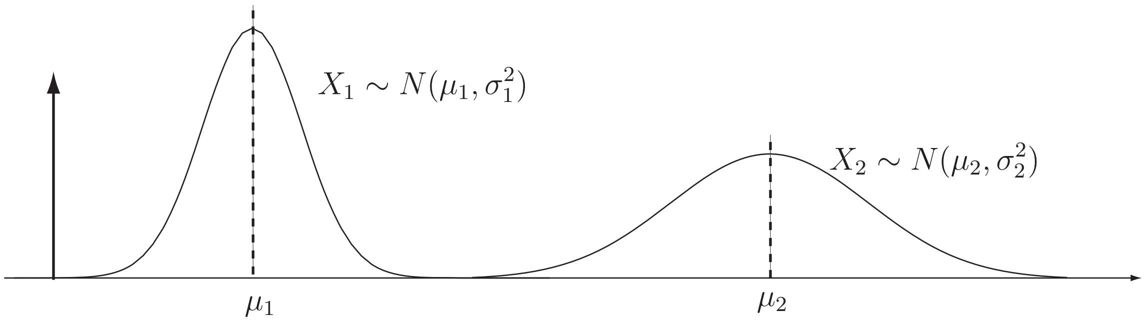


Figure 6

When we look at hypothesis testing using two means, we will be considering the difference $\mu_1 - \mu_2$ of the means and writing null hypotheses of the form

$$H_0: \quad \mu_1 - \mu_2 = \text{Value}$$

As you might expect, Value will often be zero and we will be trying to detect whether there is any statistically significant evidence of a difference between the means.

We know, from our previous work on continuous distributions (see HELM 38) that:

$$\mathsf{E}(\bar{X}_1 - \bar{X}_2) = \mathsf{E}(\bar{X}_1) - \mathsf{E}(\bar{X}_2) = \mu_1 - \mu_2$$

and that

$$\mathsf{V}(\bar{X}_1 - \bar{X}_2) = \mathsf{V}(\bar{X}_1) - \mathsf{V}(\bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

since $\bar{X}_1$ and $\bar{X}_2$ are independent. Given the assumptions made we can assert that the quantity Z defined by

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

follows the standard normal distribution $N(0, 1)$.

We are now ready to apply this formula to practical problems in which random samples of different sizes are drawn from normal populations. The conditions for the rejection of H_0 at the 5% and the 1% levels of significance are exactly the same as those previously used for single sample problems.

Example 4

A motor manufacturer wishes to replace steel suspension components by aluminium components to save weight and thereby improve performance and fuel consumption. Tensile strength tests are carried out on randomly chosen samples of two possible components before a final choice is made. The results are:

Component Number	**Sample Size**	**Mean Tensile Strength** (kg mm^{-2})	**Standard Deviation** (kg mm^{-2})
1	15	90	2.3
2	10	88	2.2

Is there any difference between the measured tensile strengths at the 5% level of significance?

Solution

The null and alternative hypotheses are:

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The null hypothesis represent the statement 'there is no difference in the tensile strengths of the two components.' The test statistic Z is calculated as:

$$\begin{aligned} Z &= \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \\ &= \frac{(90 - 88) - (0)}{\sqrt{\dfrac{2.3^2}{15} + \dfrac{2.2^2}{10}}} \\ &= \frac{2}{\sqrt{0.3527 + 0.484}} \\ &= 2.186 \end{aligned}$$

Since $2.186 > 1.96$ we conclude that, on the basis of the (limited) evidence available, there is a difference in tensile strength between the components tested. The manufacturer should carry out more comprehensive tests before making a final decision as to which component to use. The decision is a serious one with safety implications as well as economic implications. As well as carrying out more tests the manufacturer should consider the level of rejection of the null hypothesis, perhaps using 1% instead of 5%. Component 1 appears to be stronger but this may not be the case after more tests are carried out.

Task

A motor manufacturer is considering whether or not a new fuel formulation will improve the maximum power output of a particular type of engine. Tests are carried out on randomly chosen samples of the two fuels in order to inform a decision. The results are:

Fuel Type	Sample Size	Mean Maximum Power Output (bhp)	Standard Deviation (bhp)
1	20	1350	10
2	16	131	8

Is there any difference between the measured power outputs at the 5% level of significance?

Your solution

Answer

The null and alternative hypotheses are:

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The null hypothesis represent the statement 'there is no difference in the measured maximum power outputs'. The test statistic Z is calculated as:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{(135 - 131) - (0)}{\sqrt{\frac{10^2}{20} + \frac{8^2}{16}}} = \frac{4}{\sqrt{5+4}} = 1.33$$

Since $1.33 < 1.96$ we conclude that, on the basis of the (limited) evidence available, there is insufficient evidence to conclude that there is a difference in the maximum power output of the engines tested when run on the different types of fuel.

Two independent populations each with an unknown variance

Again we assume that the populations are normally distributed and use the same standard notation used previously to describe the populations and samples, namely:

Population	Sample
$X_1 \sim N(\mu_1, \sigma_1^2)$	$x_{11}, x_{12}, x_{13}, \cdots, x_{1n_1}$ with n_1 members.
$X_2 \sim N(\mu_2, \sigma_2^2)$	$x_{21}, x_{22}, x_{23}, \cdots, x_{2n_2}$ with n_2 members.

There are two distinct cases to consider. Firstly, we will assume that although the variances are unknown, they are in fact equal. Secondly, we will assume that the unknown variances are not necessarily equal.

Case (i) - Unknown but equal variances

Again, when we look at hypothesis testing using two means, we will be considering the difference $\mu_1 - \mu_2$ of the means and writing null hypotheses of the form

$$H_0: \quad \mu_1 - \mu_2 = \text{Value}$$

and again Value will often be zero and we will be trying to detect whether there is any statistically significant difference between the means.

We will take $\sigma_1^2 = \sigma_2^2 = \sigma^2$ so that in diagrammatic form the populations are:

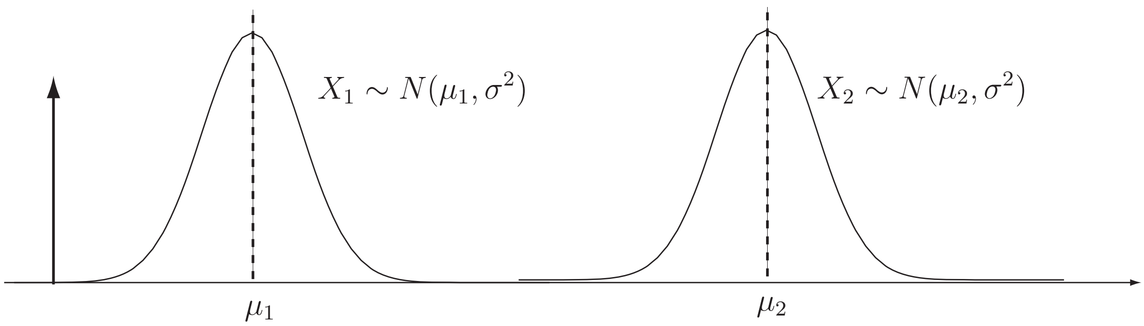


Figure 7

The results from our work on continuous distributions (see HELM 38) tell us that:

$$\mathsf{E}(\bar{X}_1 - \bar{X}_2) = \mathsf{E}(\bar{X}_1) - \mathsf{E}(\bar{X}_2) = \mu_1 - \mu_2$$

as before, and that

$$\mathsf{V}(\bar{X}_1 - \bar{X}_2) = \mathsf{V}(\bar{X}_1) - \mathsf{V}(\bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Given that we do not know the value of σ, we must estimate it. This is done by combining (or pooling) the sample variances say S_1^2 and S_2^2 for samples 1 and 2 respectively according to the formula:

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Notice that

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{(n_1 - 1)S_1^2}{n_1 + n_2 - 2} + \frac{(n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

so that you can see that S_c^2 is a weighted average of S_1^2 and S_2^2. In fact, each sample variance is weighted according to the number of degrees of freedom available. Notice also that the first sample contributes $n_1 - 1$ degrees of freedom and the second sample contributes $n_2 - 1$ degrees of freedom so that S_c^2 has $n_1 + n_2 - 2$ degrees of freedom.

Since we are estimating unknown variances, the quantity T defined by

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

will follow Student's t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

We are now ready to apply this formula to practical problems in which random samples of different sizes with unknown but equal variances are drawn from independent normal populations. The conditions for the rejection of H_0 at the 5% and the 1% levels of significance are found from tables of the t-distribution (Table 2), a copy of which is included to the end of this Workbook.

Example 5

A manufacturer of electronic equipment has developed a circuit to feed current to a particular component in a computer display screen. While the new design is cheaper to manufacture, it can only be adopted for mass production if it passes the same average current to the component. In tests involving the two circuits, the following results are obtained.

Test Number	Circuit 1 - Current (mA)	Circuit 2 - Current (mA)
1	80.1	80.7
2	82.3	81.3
3	84.1	84.6
4	82.6	81.7
5	85.3	86.3
6	81.3	84.3
7	83.2	83.7
8	81.7	84.7
9	82.2	82.8
10	81.4	84.4
11		85.2
12		84.9

On the assumption that the populations from which the samples are drawn **have equal variances**, should the manufacturer replace the old circuit design by the new one? Use the 5% level of significance.

Solution

If the average current flows are represented by μ_1 and μ_2 we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The sample means are $\bar{X}_1 = 82.42$ and $\bar{X}_2 = 83.72$.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The pooled estimate of the variance is

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9 \times 2.00 + 11 \times 2.72}{20} = 2.396$$

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{82.42 - 83.72}{\sqrt{2.396}\sqrt{\frac{1}{10} + \frac{1}{12}}} = -1.267$$

From t-tables, the critical values with 20 degrees of freedom and a two-tailed test are ± 2.086. Since $-2.086 < -1.267 < 2.086$ we conclude that we cannot reject the null hypothesis in favour of the alternative. A 95% confidence interval for the difference between the mean currents is given by $\bar{x}_1 - \bar{x}_2 \pm 2.086 \times S_c\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$. The confidence interval is $-2.683 < \mu_1 - \mu_2 < 0.083$.

Task

A manufacturer of steel cables used in the construction of suspension bridges has experimented with a new type of steel which it is hoped will result in the cables produced being stronger in the sense that they will accept greater tension loads before failure. In order to test the performance of the new cables in comparison with the old cables, samples are tested for failure under tension. The following results were obtained, the failure tensions being given in kg$\times 10^3$.

Test Number	New Cable	Original Cable
1	92.7	90.2
2	91.6	92.4
3	94.7	94.7
4	93.7	92.1
5	96.5	95.9
6	94.3	91.1
7	93.7	93.2
8	96.8	91.5
9	98.9	
10	99.9	

The cable manufacturer, on looking at health and safety legislation, decides that a 1% level of significance should be used in any statistical testing procedure adopted to distinguish between the cables. On the basis of the results given, should the manufacturer replace the old cable by the new one? You may assume that the populations from which the samples are drawn **have equal variances**.

Your solution

Answer

If the average tensions are represented by μ_1 (new cable) and μ_2 (old cable) we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 > 0$$

in order to test the hypothesis that the new cable is stronger on average than the old cable.

The sample means are $\bar{X}_1 = 95.28$ and $\bar{X}_2 = 92.64$.

The sample variances are $S_1^2 = 6.47$ and $S_2^2 = 3.14$.

The pooled estimate of the variance is

$$S_c^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9 \times 6.47 + 7 \times 3.14}{16} = 5.013$$

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{95.28 - 92.64}{\sqrt{2.239}\sqrt{\dfrac{1}{10} + \dfrac{1}{8}}} = \frac{2.64}{2.239 \times \sqrt{0.225}} = 2.486$$

Using t-distribution tables with 16 degrees of freedom, we see that the critical value at the 1% level of significance is 2.583. Since $2.486 < 2.583$ we conclude that we cannot reject the null hypothesis in favour of the alternative. However, the close result indicates that more tests should be carried out before making a final decision. At this stage the cable manufacturer should not replace the old cable by the new one on the basis of the evidence available.

Case (ii) - Unknown and unequal variances

In this case we will take $\sigma_1^2 \neq \sigma_2^2$ so that in diagrammatic form the populations may be represented as shown below.

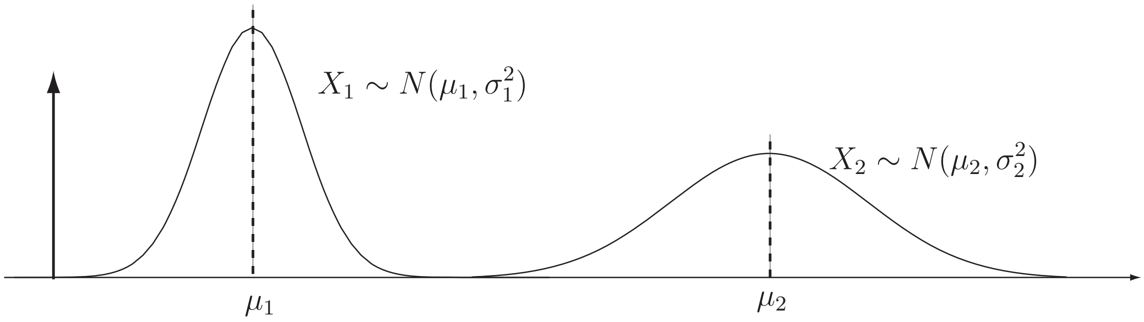


Figure 8

Again, when we look at hypothesis testing using two means, we will be considering the difference $\mu_1 - \mu_2$ of the means and writing null hypotheses of the form

$H_0: \quad \mu_1 - \mu_2 = \text{Value}$

and again Value will often be zero and we will be trying to detect whether there is any statistically significant difference between the means.

In the case where we assume unequal variances, there is no exact statistic which we can use to test the validity or otherwise of the null hypothesis $H_0: \quad \mu_1 - \mu_2 = \text{Value}$. However, the following approximation in Key Point 1 allows us to overcome this problem.

Key Point 1

Provided that the null hypothesis is true, the statistic

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_c\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

will *approximately* follow Student's distribution with the number of degrees of freedom given by the expression:

$$\nu = \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{S_1^2}{n_1}\right)^2}{n_1 + 1} + \dfrac{\left(\dfrac{S_2^2}{n_2}\right)^2}{n_2 + 1}} - 2$$

Essentially, this means that the actual test procedure is similar to that used previously but with T and the number of degrees of freedom ν calculated using the above formulae.

We are now ready to apply these formulae to practical problems in which random samples of different sizes with unknown and unequal variances are drawn from independent normal populations. We will illustrate the test procedure by reworking an Example and Task done previously but we will assume unequal rather than equal variances.

This next Example is a repeat of Example 5 but here assuming unequal variances.

Example 6

A manufacturer of electronic equipment has developed a circuit to feed current to a particular component in a computer display screen. While the new design is cheaper to manufacture, it can only be adopted for mass production if it passes the same average current to the component. In tests involving the two circuits, the results are obtained are:

Test Number	Circuit 1 - Current (mA)	Circuit 2 - Current (mA)
1	80.1	80.7
2	82.3	81.3
3	84.1	84.6
4	82.6	81.7
5	85.3	86.3
6	81.3	84.3
7	83.2	83.7
8	81.7	84.7
9	82.2	82.8
10	81.4	84.4
11		85.2
12		84.9

On the assumption that the populations from which the samples are drawn **do not have equal variances**, should the manufacturer replace the old circuit design by the new one? Use the 5% level of significance.

Solution

If the average current flows are represented by μ_1 and μ_2 we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 \neq 0$$

The sample means are $\bar{X}_1 = 82.42$ and $\bar{X}_2 = 83.72$.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{82.42 - 83.72}{\sqrt{\frac{2.00}{10} + \frac{2.72}{12}}} = -\frac{1.3}{\sqrt{0.427}} = -1.990$$

Solution (contd.)

The number of degrees of freedom is given by

$$\begin{aligned}
\nu &= \frac{\left(\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{S_1^2}{n_1}\right)^2}{n_1+1}+\dfrac{\left(\dfrac{S_2^2}{n_2}\right)^2}{n_2+1}}-2 \\
&= \frac{\left(\dfrac{2.00}{10}+\dfrac{2.72}{12}\right)^2}{\dfrac{(2.00/10)^2}{11}+\dfrac{(2.72/12)^2}{13}}-2=\frac{0.182}{0.004+0.004}-2\approx 21
\end{aligned}$$

From t-tables, the critical values (two-tailed test, 5% level of significance) are ± 2.080. Since $-2.080 < -1.990 < 2.080$ we conclude that there is insufficient evidence to reject the null hypothesis in favour of the alternative at the 5% level of significance.

This next Task is a repeat of the Task on page 25 but assuming unequal variances.

Task

A manufacturer of steel cables used in the construction of suspension bridges has experimented with a new type of steel which it is hoped will result in the cables produced being stronger in the sense that they will accept greater tension loads before failure. In order to test the performance of the new cables in comparison with the old cables, samples are tested for failure under tension. The results obtained are given below where the failure tensions are given in kg$\times 10^3$.

Test Number	New Cable	Original Cable
1	92.7	90.2
2	91.6	92.4
3	94.7	94.7
4	93.7	92.1
5	96.5	95.9
6	94.3	91.1
7	93.7	93.2
8	96.8	91.5
9	98.9	
10	99.9	

The cable manufacturer, on looking at health and safety legislation, decides that a 1% level of significance should be used in any statistical testing procedure adopted to distinguish between the cables. On the basis of the results given and assuming that the populations from which the samples are drawn **do not have equal variances**, should the manufacturer replace the old cable by the new one?

Your solution

Answer

If the average tensions are represented by μ_1 (new cable) and μ_2 (old cable), we form the hypotheses

$$H_0: \quad \mu_1 - \mu_2 = 0 \qquad H_1: \quad \mu_1 - \mu_2 > 0$$

in order to test the hypothesis that the new cable is stronger on average than the old cable.

The sample means are $\bar{X}_1 = 95.28$ and $\bar{X}_2 = 92.64$.

The sample variances are $S_1^2 = 6.47$ and $S_2^2 = 3.14$.

The test statistic is

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{95.28 - 92.64}{\sqrt{\frac{6.47}{10} + \frac{3.14}{8}}} = \frac{2.64}{\sqrt{1.017}} = 2.589$$

The number of degrees of freedom is given by

$$\nu = \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{S_1^2}{n_1}\right)^2}{n_1+1} + \dfrac{\left(\dfrac{S_2^2}{n_2}\right)^2}{n_2+1}} - 2 = \frac{\left(\dfrac{6.47}{10} + \dfrac{3.14}{8}\right)^2}{\dfrac{(6.47/10)^2}{11} + \dfrac{(3.14/8)^2}{9}} - 2 = \frac{1.081}{0.038 + 0.017} - 2 \approx 4$$

Using t-distribution tables with 18 degrees of freedom, we see that the critical value at the 1% level of significance is 2.552. Since $2.589 < 2.552$ we conclude that we reject the null hypothesis in favour of the alternative. Notice that the result could still be considered marginal. The cable manufacturer should exercise caution if the old cable is replaced by the new one on the basis of the evidence available.

The F-test

In the tests above, we distinguished between the cases of equal and unequal variances of samples chosen from independent normal populations. As you have seen, the analysis changes according to the assumptions made, conclusions reached and recommendations made - accepting or rejecting a null hypothesis for example - may also change. In view of this, we may wish to test in order to decide whether the assumption that the variances σ_1^2 and σ_2^2 of the independent normal populations shown in the diagram below, may be regarded as equal.

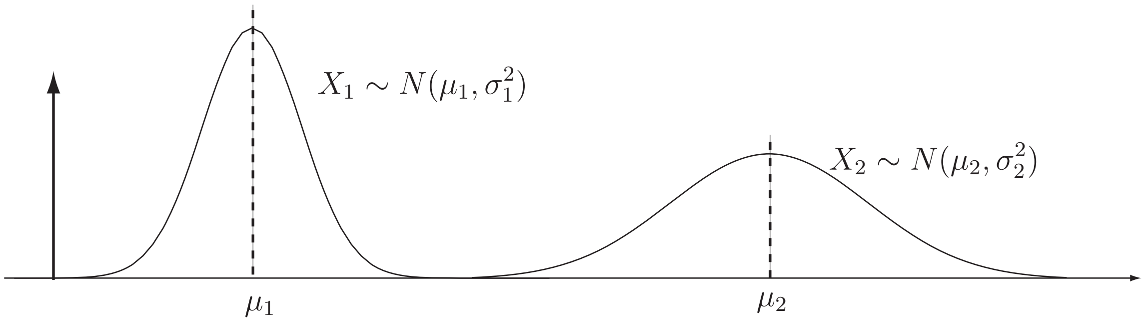


Figure 9

Essentially, we will test the null hypothesis

$$H_0: \quad \sigma_1^2 = \sigma_2^2$$

against one of the alternatives

$$H_1: \quad \sigma_1^2 \neq \sigma_2^2 \qquad H_1: \quad \sigma_1^2 > \sigma_2^2 \qquad H_1: \quad \sigma_1^2 < \sigma_2^2$$

In order to do this, we use the F-distribution. The hypothesis test for the equality of two variances σ_1^2 and σ_2^2 is encapsulated in the following Key Point.

Key Point 2

Consider a random sample of size n_1 taken from a normal population with mean μ_1 and variance σ_1^2 and a random sample of size n_1 taken from a second normal population with mean μ_2 and variance σ_2^2. Denote the respective sample variances by S_1^2 and S_2^2 and assume that the populations are independent. The ratio

$$F = \frac{S_1^2}{\sigma_1^2} \Big/ \frac{S_2^2}{\sigma_2^2}$$

follows an F distribution in which the numerator has $n_1 - 1$ degrees of freedom and the denominator has $n_2 - 1$ degrees of freedom.

Note that if the null hypothesis $H_0: \quad \sigma_1^2 = \sigma_2^2$ is true, then the value of F reduces to the ratio of the sample variances and that in this case

$$F = \frac{S_1^2}{S_2^2}$$

Note

Recall that if a random sample of size n_1 is taken from a normal population with mean μ_1 and variance σ_1^2 and if the sample variance is denoted by S_1^2, the random variable

$$X_1^2 = \frac{(n_1 - 1)S_1^2}{\sigma_1^2}$$

has a χ^2 distribution with $n_1 - 1$ degrees of freedom. Similarly, if a random sample of size n_2 is taken from a normal population with mean μ_2 and variance σ_2^2 and if the sample variance is denoted by S_2^2, the random variable

$$X_2^2 = \frac{(n_2 - 1)S_2^2}{\sigma_2^2}$$

has a χ^2 distribution with $n_2 - 1$ degrees of freedom. This means that the ratio

$$F = \frac{S_1^2}{\sigma_1^2} \Big/ \frac{S_2^2}{\sigma_2^2}$$

is a ratio of χ^2 random variables with $n_1 - 1$ degrees of freedom in the numerator and $n_2 - 1$ degrees of freedom in the denominator. Under the null hypothesis

$$H_0: \quad \sigma_1^2 = \sigma_2^2$$

we know that the expression for F reduces to

$$F = \frac{S_1^2}{S_2^2}$$

and we say that F has an F-distribution with $n_1 - 1$ degrees of freedom in the numerator and $n_2 - 1$ degrees of freedom in the denominator. This distribution is denoted by

$$F_{n_1-1, n_2-1}$$

and some tabulated values are given in Tables 3 and 4 at the end of this Workbook.

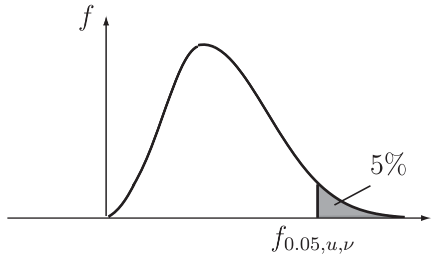
If you check Tables 3 and 4, you will find that only right-tail values are given. The left-tail values are calculated by using the following formula:

$$f_{1-\alpha,\ n_1-1,\ n_2-1} = \frac{1}{f_{\alpha,\ n_2-1,\ n_1-1}}$$

Note the reversal in the order in which the expressions for the number of degrees of freedom occur.

Example 7

The following is an extract from the F-distribution tables (5% tail) given at the end of this Workbook.



	Degrees of Freedom for the Numerator (u)														
ν	1	2	3	4	5	6	7	8	9	10	20	30	40	60	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	248.0	250.1	251.1	252.2	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.45	19.46	19.47	19.48	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.66	8.62	8.59	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.80	5.75	5.72	5.69	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.56	4.53	4.46	4.43	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.87	3.81	3.77	3.74	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.44	3.38	3.34	3.30	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.15	3.08	3.04	3.01	2.93

Figure 10

Write down or calculate as appropriate, the following values of F from the table:

Right-tail Values	Left-tail Values
$f_{0.05,4,3}$	$f_{0.95,4,3}$
$f_{0.05,8,2}$	$f_{0.95,8,2}$
$f_{0.05,7,8}$	$f_{0.95,7,8}$

Solution

The right-tail values are read directly from the tables. The left-tail values are calculated using the formula given above.

Right-tail Values	Left-tail Values
$f_{0.05,4,3} = 9.12$	$f_{0.95,4,3} = \frac{1}{f_{0.05,3,4}} = \frac{1}{6.59} = 0.152$
$f_{0.05,8,2} = 19.37$	$f_{0.95,8,2} = \frac{1}{f_{0.05,2,8}} = \frac{1}{4.46} = 0.224$
$f_{0.05,7,8} = 3.50$	$f_{0.95,7,8} = \frac{1}{f_{0.05,8,7}} = \frac{1}{3.73} = 0.268$

Task

Write down or calculate as appropriate, the following values of F from the tables given at the end of this Workbook.

Right-tail Values	Left-tail Values
$f_{0.05,10,20}$	$f_{0.95,10,20}$
$f_{0.05,5,30}$	$f_{0.95,5,30}$
$f_{0.05,20,7}$	$f_{0.95,20,7}$
$f_{0.025,10,10}$	$f_{0.975,10,10}$
$f_{0.025,8,30}$	$f_{0.975,8,30}$
$f_{0.025,20,30}$	$f_{0.975,20,30}$

Your solution

Right-tail Values	Left-tail Values
$f_{0.05,10,20} =$	$f_{0.95,10,20} =$
$f_{0.05,5,30} =$	$f_{0.95,5,30} =$
$f_{0.05,20,7} =$	$f_{0.95,20,7} =$
$f_{0.025,10,10} =$	$f_{0.975,10,10} =$
$f_{0.025,8,30} =$	$f_{0.975,8,30} =$
$f_{0.025,20,30} =$	$f_{0.975,20,30} =$

Answer

Right-tail Values	Left-tail Values
$f_{0.05,10,20} = 2.35$	$f_{0.95,10,20} = \frac{1}{f_{0.05,20,10}} = \frac{1}{2.77} = 0.361$
$f_{0.05,5,30} = 2.53$	$f_{0.95,5,30} = \frac{1}{f_{0.05,30,5}} = \frac{1}{4.53} = 0.221$
$f_{0.05,20,7} = 3.44$	$f_{0.95,20,7} = \frac{1}{f_{0.05,7,20}} = \frac{1}{2.51} = 0.398$
$f_{0.025,10,10} = 3.72$	$f_{0.975,10,10} = \frac{1}{f_{0.025,10,10}} = \frac{1}{3.72} = 0.269$
$f_{0.025,8,30} = 2.65$	$f_{0.975,8,30} = \frac{1}{f_{0.025,30,8}} = \frac{1}{3.89} = 0.257$
$f_{0.025,20,30} = 2.20$	$f_{0.975,20,30} = \frac{1}{f_{0.025,30,20}} = \frac{1}{2.35} = 0.426$

We are now in a position to use the F-test to solve engineering problems. The application of the F-test will be illustrated by using the data given in a previous worked example in order to determine whether the assumption of equal variability in the samples used is realistic.

This next Example was met as Example 5 (page 24). Here we test one of the underlying assumptions.

Example 8

A manufacturer of electronic equipment has developed a circuit to feed current to a particular component in a computer display screen. While the new design is cheaper to manufacture, it can only be adopted for mass production if it passes the same average current to the component. In tests involving the two circuits, the results obtained are

Test Number	Circuit 1 - Current (mA)	Circuit 2 - Current (mA)
1	80.1	80.7
2	82.3	81.3
3	84.1	84.6
4	82.6	81.7
5	85.3	86.3
6	81.3	84.3
7	83.2	83.7
8	81.7	84.7
9	82.2	82.8
10	81.4	84.4
11		85.2
12		84.9

In Example 5 we worked on the assumption that the populations from which the samples are drawn have equal variances. Is this assumption valid at the 5% level of significance?

Note that the manufacturer may also be interested in knowing whether the variances are equal as well as the means. We shall not address that problem here but it can be argued that equality of variances will facilitate consistent performance from the components.

Solution

We form the hypotheses

$$H_0: \quad \sigma_1^2 = \sigma_2^2 \qquad H_1: \quad \sigma_1^2 \neq \sigma_2^2$$

and perform a two-tailed test.

The sample variances are $S_1^2 = 2.00$ and $S_2^2 = 2.72$.

The test statistic is

$$F = \frac{S_1^2}{S_2^2} = \frac{2.00}{2.72} = 0.735$$

which has an F-distribution with 9 degrees of freedom in the numerator and 11 degrees of freedom in the denominator.

Solution (contd.)

We require two 2.5% tails, that is we require right-tail $f_{0.025,9,11} = 3.59$ and left-tail $f_{0.975,9,11}$. The latter may be approximated as follows:

$$f_{0.975,9,11} = \frac{1}{f_{0.025,11,9}} \approx \frac{\left(\frac{1}{11}-\frac{1}{20}\right)\frac{1}{f_{0.025,10,9}} + \left(\frac{1}{10}-\frac{1}{11}\right)\frac{1}{f_{0.025,20,9}}}{\left(\frac{1}{10}-\frac{1}{20}\right)}$$

$$\approx \frac{\frac{0.040909}{3.96} + \frac{0.009091}{3.67}}{0.05}$$

$$\approx \frac{0.81818}{3.96} + \frac{0.18182}{3.67} = 0.256$$

Since $0.256 < 0.735 < 3.59$ we conclude that we cannot reject the null hypothesis in favour of the alternative at the 5% level of significance. The evidence supports the conclusion that the samples have equal variability.

Note that we can adopt the rule (many statisticians do this) of always dividing the larger S^2 value by the smaller S^2 value so that you only need to look up right tail values.

This next Task was first met on page 25. Here we test one of the underlying assumptions.

Task

A manufacturer of steel cables used in the construction of suspension bridges has experimented with a new type of steel which it is hoped will result in the cables produced being stronger in the sense that they will accept greater tension loads before failure. In order to test the performance of the new cables in comparison with the old cables, samples are tested for failure under tension. The results obtained are as follows, where the failure tensions are given in tonnes.

Test Number	New steel cable tension	Old steel cable tension
1	80.1	80.7
2	82.3	81.3
3	84.1	84.6
4	82.6	81.7
5	85.3	86.3
6	81.3	84.3
7	83.2	83.7
8	81.7	84.7
9	82.2	82.8
10	81.4	84.4
11		85.2
12		84.9

Last time we assumed that the populations from which the samples are drawn did not have equal variances. Is this assumption valid at the 5% level of significance?

Your solution

Answer

We form the hypotheses

$$H_0: \quad \sigma_1^2 = \sigma_2^2 \qquad H_1: \quad \sigma_1^2 \neq \sigma_2^2$$

and perform a two-tailed test.

The sample variances are $S_1^2 = 6.47$ and $S_2^2 = 3.14$.

The test statistic is

$$F = \frac{S_1^2}{S_2^2} = \frac{6.47}{3.14} = 2.061$$

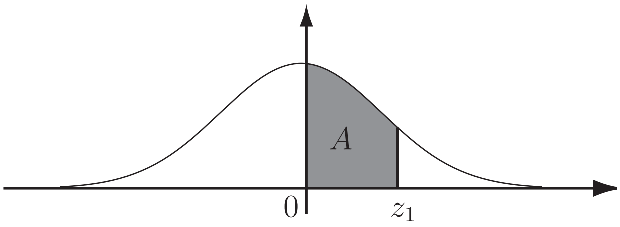
which has an F-distribution with 9 degrees of freedom in the numerator and 7 degrees of freedom in the denominator. We require two 2.5% tails. That is, we require right-tail $f_{0.025,9,7} = 4.42$ and left-tail $f_{0.975,9,7}$ which may be calculated as

$$f_{0.975,9,7} = \frac{1}{f_{0.025,7,9}} = \frac{1}{4.20} = 0.238$$

Since $0.238 < 2.061 < 4.82$ we conclude that we cannot reject the null hypothesis in favour of the alternative at the 5% level of significance. The evidence does not support the conclusion that the populations have unequal variances.

Table 1: The Normal Probability Integral

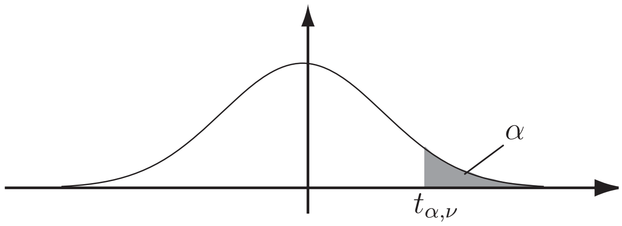
The area is denoted by A and is measured from the mean $z = 0$ to any ordinate $z = z_1$.



$Z = \frac{X-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	0040	0080	0120	0159	0199	0239	0279	0319	0359
0.1	.0398	0438	0478	0517	0557	0596	0636	0657	0714	0753
0.2	.0793	0832	0871	0910	0948	0987	1026	1064	1103	1141
0.3	.1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
0.4	.1554	1591	1628	1664	1700	1736	1772	1808	1844	1879
0.5	.1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
0.6	.2257	2291	2324	2357	2389	2422	2454	2486	2518	2549
0.7	.2530	2611	2642	2673	2704	2734	2764	2794	2823	2852
0.8	.2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
0.9	.3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	.3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	.3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	.3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	.4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	.4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	.4332	4345	4357	4370	4382	4394	4406	4418	4430	4441
1.6	.4452	4463	4474	4485	4495	4505	4515	4525	4535	4545
1.7	.4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	.4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	.4713	4719	4726	4732	4738	4744	4750	4756	4762	4767
2.0	.4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	.4621	4826	4830	4835	4838	4842	4846	4850	4854	4857
2.2	.4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	.4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	.4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	.4938	4940	4941	4943	4945	4946	4948	4949	4951	4952
2.6	.4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	.4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	.4974	4975	4976	4977	4977	4978	4979	4980	4980	4981
2.9	.4981	4982	4983	4983	4984	4984	4985	4985	4986	4986
3.0	.4986	4987	4987	4988	4988	4989	4989	4989	4990	4990
3.1	.4990	4991	4991	4991	4992	4992	4992	4992	4993	4993
3.2	.4993	4994	4994	4994	4994	4994	4994	4995	4995	4995
3.3	.4995	4995	4995	4996	4996	4996	4996	4996	4996	4997
3.4	.4997	4997	4997	4997	4997	4997	4997	4997	4997	4998
3.5	.4998	4998	4998	4998	4998	4998	4998	4998	4998	4998
3.6	.4998	4998	4999	4999	4999	4999	4999	4999	4999	4999
3.7	.4999	4999	4999	4999	4999	4999	4999	4999	4999	4999
3.8	.4999	4999	4999	4999	4999	4999	4999	4999	4999	4999
3.9	.4999	4999	4999	4999	4999	4999	4999	4999	4999	4999

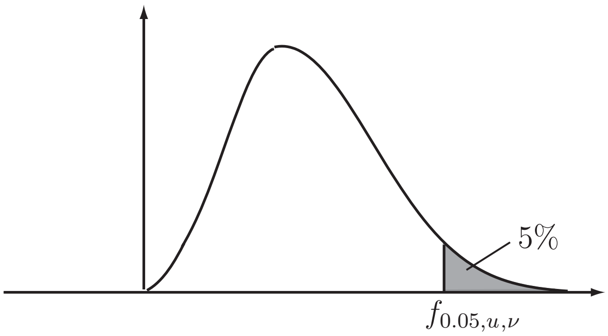
Note that some text books give the final line entries as 0.5 rather than 0.4999.
In these workbooks we shall use 0.4999.

Table 2: Percentage Points of the Students t-distribution



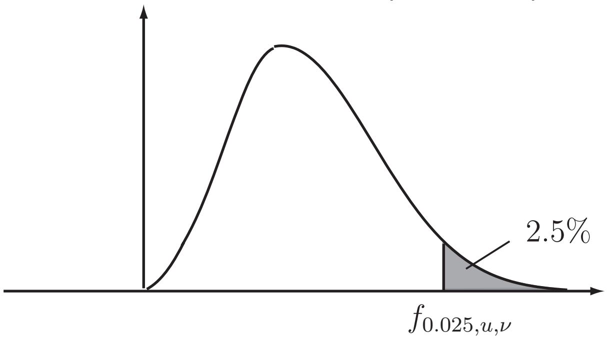
α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
ν										
1	.325	1.000	3.078	6.314	12.706	31.825	63.657	127.32	318.31	636.62
2	.289	.816	1.886	2.902	4.303	6.965	9.925	14.089	23.326	31.598
3	.277	.765	1.638	2.353	3.182	4.514	5.841	7.453	10.213	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.487
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Table 3: Percentage Points of the F-Distribution (5% tail)



	Degrees of Freedom for the Numerator (u)														
ν	1	2	3	4	5	6	7	8	9	10	20	30	40	60	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	248.0	250.1	251.1	252.2	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.45	19.46	19.47	19.48	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.66	8.62	8.59	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.80	5.75	5.72	5.69	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.56	4.53	4.46	4.43	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.87	3.81	3.77	3.74	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.44	3.38	3.34	3.30	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.15	3.08	3.04	3.01	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	2.94	2.86	2.83	2.79	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.77	2.70	2.66	2.62	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.65	2.57	2.53	2.49	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.54	2.47	2.43	2.38	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.46	2.38	2.34	2.30	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.39	2.31	2.27	2.22	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.33	2.25	2.20	2.16	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.28	2.19	2.15	2.11	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.23	2.15	2.10	2.06	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.19	2.11	2.06	2.02	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.16	2.07	2.03	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.12	2.04	1.99	1.95	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.10	2.01	1.96	1.92	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.07	1.98	1.94	1.89	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.05	1.96	1.91	1.86	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.03	1.94	1.89	1.84	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.01	1.92	1.87	1.82	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	1.99	1.90	1.85	1.80	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	1.97	1.88	1.84	1.79	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	1.96	1.87	1.82	1.77	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	1.94	1.85	1.81	1.75	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	1.93	1.84	1.79	1.74	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	1.84	1.74	1.69	1.64	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.75	1.65	1.59	1.53	1.39
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.57	1.46	1.39	3.32	1.00

Table 4: Percentage Points of the F-Distribution (2.5% tail)



	Degrees of Freedom for the Numerator (u)														
ν	1	2	3	4	5	6	7	8	9	10	20	30	40	60	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	993.1	1001	1006	1010	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.45	39.46	39.47	39.48	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.17	14.08	14.04	13.99	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.56	8.46	8.41	8.36	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.33	6.23	6.18	6.12	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.17	5.07	5.01	4.96	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.75	4.47	4.36	4.31	4.25	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.00	3.89	3.84	3.78	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.67	3.56	3.51	3.45	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.42	3.31	3.26	3.20	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.23	3.12	3.06	3.00	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.07	2.96	2.91	2.85	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	2.95	2.84	2.78	2.72	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	2.84	2.73	2.67	2.61	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.76	2.64	2.59	2.52	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.32	3.12	3.05	2.99	2.68	2.57	2.51	2.45	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.62	2.50	2.44	2.38	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.56	2.44	2.38	2.32	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.51	2.39	2.33	2.27	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.46	2.35	2.29	2.22	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.42	2.31	2.25	2.18	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.39	2.27	2.21	2.14	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.36	2.24	2.18	2.11	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.33	2.21	2.15	2.08	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.30	2.18	2.12	2.05	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.28	2.16	2.09	2.03	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.25	2.13	2.07	2.00	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.23	2.11	2.05	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.21	2.09	2.03	1.96	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.20	2.07	2.01	1.94	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.07	1.94	1.88	1.80	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	1.94	1.82	1.74	1.67	1.48
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.71	1.57	1.48	1.39	1.00